

## Incompressible Quantum Liquids and New Conservation Laws

Alexander Seidel,<sup>1,2</sup> Henry Fu,<sup>1</sup> Dung-Hai Lee,<sup>1,2,3</sup> Jon Magne Leinaas,<sup>4</sup> and Joel Moore<sup>1,2</sup>

<sup>1</sup>*Department of Physics, University of California at Berkeley, Berkeley, California 94720, USA*

<sup>2</sup>*Material Science Division, Lawrence Berkeley National Laboratory, Berkeley, California 94720, USA*

<sup>3</sup>*Center for Advanced Study, Tsinghua University, Beijing 100084, China*

<sup>4</sup>*Department of Physics, University of Oslo, P.O. Box 1048 Blindern, 0316 Oslo, Norway*

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In this Letter, we investigate a class of Hamiltonians which, in addition to the usual center-of-mass momentum conservation, also have center-of-mass position conservation. We find that, regardless of the particle statistics, the energy spectrum is at least  $q$ -fold degenerate when the filling factor is  $p/q$ , where  $p$  and  $q$  are coprime integers. Interestingly, the simplest Hamiltonian respecting this type of symmetry encapsulates two prominent examples of novel states of matter, namely, the fractional quantum Hall liquid and the quantum dimer liquid. We discuss the relevance of this class of Hamiltonian to the search for featureless Mott insulators.

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In the twentieth century, the basic notions of “symmetry” and “order” have set a paradigm that gave rise to a lasting cross-fertilization between various branches of physics. Landau [1] first put forth the idea that, in the process of ordering, a system also becomes less symmetric. This phenomenon is commonly known as “symmetry breaking” and underlies our general understanding of order in a wide variety of systems. Ordering is such a ubiquitous tendency of nature that it is difficult to give an example where a system remains unordered to the lowest temperatures. In recent years, condensed matter physicists have been asking whether the electrons in a solid can remain unordered at absolute zero temperature.

Following Anderson’s proposal of the “spin liquid” [2], a novel magnetic state with no order, for the past 20 years condensed matter physicists have expended tremendous effort searching for unordered electronic states in solids [3–5]. Recently, there has been significant theoretical progress in demonstrating the stability of some examples of such states and, hence, proving that they are, in principle, possible [6–8]. In addition, it has been shown that there is a strong tie between such states and the phenomenon of “quantum number fractionalization” [9]. Concurrent with these efforts, there has been an extensive pursuit for microscopic models exhibiting these properties [10–15]. The traditional guideline for such pursuits has been “frustration” (see, e.g., [16–18]). In this Letter, we adopt a different viewpoint and investigate whether quantum disordered fermion or boson states can arise from unusual conservation laws.

It is commonly perceived that one of the most difficult situations for a system to be free of symmetry breaking is in fractionally filled Mott insulators. For bosons, the filling factor is just the average number of particles per crystalline unit cell, and, for electrons, it is half that quantity. In this Letter, we view any nondisordered insulating state as a Mott insulator if the insulating property cannot be attributed to the Pauli exclusion principle. Whether a fraction-

ally filled Mott insulator can be “featureless” (i.e., *unordered*) has been a central issue of debate in recent years. So far, all known such systems possess order [19]. For example, a half-filled square lattice of boson atoms is only known to be Mott insulating when the atoms localize in a checkerboard pattern and, hence, break the symmetry of the original lattice [20].

Physicists have only recently begun to understand the reason why symmetry breaking usually accompanies Mott insulators. This progress is due to generalizations [21,22] of an earlier theorem by Lieb, Schultz, and Mattis [23]. In particular, Oshikawa [21] has argued that, when a fractionally filled Mott insulator has an energy gap between its ground state(s) and excited states, it must have ground state degeneracy. In particular, if the filling factor is  $p/q$ , the ground state must be at least  $q$ -fold degenerate. Empirically, this degeneracy is always achieved by breaking symmetry. In the boson Mott insulator example given above, a twofold degeneracy arises because there are two distinct, complementary, checkerboard-ordered patterns. For the time being, we will use the phrase “Mott insulator” to refer to a fractionally filled Mott insulator with an energy gap.

Symmetry breaking is one way to produce the degeneracy required by Oshikawa’s result. However, if there were a way to produce this degeneracy independently of symmetry breaking, it would provide a candidate route to finding featureless Mott insulators. In the following, we show that imposing center-of-mass (c.m.) momentum and position conservation is one such way.

*Theorem: Simultaneous conservation of center-of-mass momentum and position guarantees  $q$ -fold degeneracy of the energy spectrum.*—Consider a  $D$ -dimensional lattice. We impose periodic boundary conditions in a certain direction, say, the  $x$  direction, of the lattice. Let  $L$  be the spatial period in this direction and  $T$  be the operator of translation by one lattice constant in the  $x$  direction. The “cross section”  $C$  of the lattice is defined such that the total

number of unit cells is given by  $CL$ . Following Oshikawa, let us now consider filling factor  $\nu = p/q$  and  $C$  relatively prime to  $q$ . In this case, Oshikawa's argument tells us that the ground state is at least  $q$ -fold degenerate if there is an energy gap.

The (exponentiated) c.m. position in the  $x$  direction modulo  $L$  is given by

$$U = \exp\left(i\frac{2\pi}{L}\sum_{\vec{r}} x C_{\vec{r}}^{\dagger} C_{\vec{r}}\right). \quad (1)$$

The simultaneous conservation of the c.m. momentum and position implies  $[H, U] = [H, T] = 0$ . Because of this, we can choose the eigenstates of  $H$  to be simultaneous eigenstates of  $U$ . In addition, since

$$T^{-1}UT = \exp\left[i\frac{2\pi}{L}\sum_{\vec{r}} (x+1)C_{\vec{r}}^{\dagger} C_{\vec{r}}\right] = e^{i2\pi Cp/q}U, \quad (2)$$

we obtain

$$UT = e^{i2\pi Cp/q}TU. \quad (3)$$

Now we show that  $q$  consecutive actions of  $T$  on any energy eigenstate generates  $q$  degenerate orthogonal states differentiated by their eigenvalue with respect to  $U$ . Let the eigenstate in question be denoted by  $|e^{i\phi}, E\rangle$ , where  $e^{i\phi}$  is the eigenvalue with respect to  $U$  and  $E$  is the energy eigenvalue. Because the translation operator commutes with the Hamiltonian, the state  $T|e^{i\phi}, E\rangle$  is also an eigenstate of  $H$  with the same energy. However, its eigenvalue with respect to  $U$  is

$$\begin{aligned} UT|e^{i\phi}, E\rangle &= e^{i2\pi Cp/q}TU|e^{i\phi}, E\rangle \\ &= (e^{i2\pi Cp/q}e^{i\phi})T|e^{i\phi}, E\rangle. \end{aligned} \quad (4)$$

The different eigenvalue with respect to  $U$  implies that the two energy eigenstates  $|e^{i\phi}, E\rangle$  and  $T|e^{i\phi}, E\rangle$  are orthogonal. We can perform this operation  $q$  times before we arrive at a state with the original  $U$  eigenvalue. Hence, there is at least a  $q$ -fold degeneracy.

Motivated by the above observations, we would like to analyze Hamiltonians featuring c.m. position and momentum conservation to determine if they lead to featureless Mott insulators. The simplest nontrivial such Hamiltonian is

$$H = \sum_{\mathbf{R}, \mathbf{x}, \mathbf{y}} g(\mathbf{x}, \mathbf{y}) C_{\mathbf{R}+\mathbf{x}}^{\dagger} C_{\mathbf{R}-\mathbf{x}}^{\dagger} C_{\mathbf{R}-\mathbf{y}} C_{\mathbf{R}+\mathbf{y}}. \quad (5)$$

This Hamiltonian hops a pair of particles while conserving their c.m. position. In general, it is quite difficult to solve for the eigenstates of this type of Hamiltonian. However, for a special one-dimensional case, some insight can be gained due to a connection with a well-studied problem in physics, the fractional quantum Hall effect. To be specific, we consider the Trugman-Kivelson (TK) Hamiltonian [24], which has Laughlin's  $1/3$  wave function [25] as the ground state. When studied on a torus with dimensions  $L_x$  and  $L_y$ , the TK Hamiltonian can be written in terms of the

creation or annihilation operators of the lowest Landau level as follows [26]:

$$\begin{aligned} H &= \sum_{R,x,y} f^*(x)f(y)C_{R+x}^{\dagger}C_{R-x}^{\dagger}C_{R-y}C_{R+y}, \\ f(x) &= \kappa^{3/2}\sum_n (x-nL)e^{-\kappa^2(x-nL)^2}. \end{aligned} \quad (6)$$

Here

$$L = L_x L_y / 2\pi l_B^2, \quad \kappa = 2\pi l_B / L_y. \quad (7)$$

In the above,  $l_B = \sqrt{\hbar c/eB}$  is the magnetic length. This Hamiltonian acts upon a system of spinless fermions on a ring of  $L$  sites at filling factor  $1/3$ .

One important consequence of the connection between Eq. (6) and the quantum Hall effect on a torus is an unusual property of Eq. (6), which we call duality. By performing Fourier transforms, it is simple to show that, when written in terms of operators that create and annihilate particles in a fixed momentum state, Eq. (6) reads

$$H = \sum_{Q,k,q} \tilde{f}^*(k)\tilde{f}(q)C_{Q+k}^{\dagger}C_{Q-k}^{\dagger}C_{Q-q}C_{Q+q}, \quad (8)$$

where  $\tilde{f}(k) = \sum_x e^{i2kx}f(x)$ . For the  $f$  used in Eq. (6),  $\tilde{f}$  has the same form as  $f$ . This implies that the Hamiltonian in the momentum space is also described by Eq. (6) except  $\kappa \rightarrow \frac{2\pi}{\kappa L}$ . The factor  $2\pi/L$  arises from the lattice constant in the reciprocal space. For each  $L$ , there is a special  $\kappa$  value ( $\kappa^* = \sqrt{2\pi/L}$ ) for which the real space and momentum space  $\kappa$  are the same. The duality implies that energy spectra at  $\kappa$  and  $2\pi/\kappa L$  are identical. From the perspective of the quantum Hall liquid on a torus, the duality merely signifies that interchanging  $L_x$  and  $L_y$  leads to the same physical system.

Analyzing the Hamiltonian equation (6), we find that, for all practical purposes, the ground state is a featureless Mott insulator when  $\kappa$  is small. In addition, we find the surprising fact that, on a torus with finite circumference in one direction (but infinite circumference in the other), the Laughlin liquid has nonvanishing density wave order and is *adiabatically connected* to a Wigner crystal.

First, we focus on the large  $\kappa$  limit. In the large  $\kappa$  limit, we can expand the Hamiltonian in the parameter  $e^{-\kappa^2}$ . As a result, Eq. (6) is accurately approximated by

$$H = \sum_{i=1}^L [f(1/2)^2 n_{i+1} n_i + f(1)^2 n_{i+2} n_i], \quad (9)$$

where  $n_i = C_i^{\dagger} C_i$ . This Hamiltonian imposes energy penalties for having nearest-neighbor and next-nearest-neighbor particles. Its ground states are "Wigner crystals" where one out of every three lattice sites is occupied. Clearly, at  $\nu = 1/3$  there are three such ground states. These ground states possess density wave order (and, hence, are not featureless).

To quantify the degree of crystalline order, we introduce the order parameter  $\mathcal{O}$  associated with a spatial period of three lattice constants:

$$\mathcal{O} = \frac{1}{N} \sum_{j=1}^L e^{i(2\pi/3)j} \langle n_j \rangle. \quad (10)$$

In the above,  $N$  is the total particle number, and  $\langle n_j \rangle$  is the expectation value of the site occupation in (any one of) the ground state(s).  $\mathcal{O}$  is normalized such that its modulus becomes unity in the extreme crystalline limit. In Fig. 1(a), we plot  $\mathcal{O}$  as a function of  $\kappa$ , evaluated numerically for system sizes  $L = 6, 9, \dots, 24$ . For  $\kappa \geq 1.5$ , we obtain  $\mathcal{O} \approx 1$ , implying that the ground state is nearly a perfect crystal. As expected, as  $\kappa$  decreases the order parameter decreases and practically vanishes below  $\kappa \approx 0.5$  [27]. This finding raises an important question: Is the regime below  $\kappa = 0.5$  a featureless Mott insulator? The weak size dependence of the crystalline order parameter near  $\kappa = 0.5$  is inconsistent with the existence of a continuous phase transition, which is required if the state below  $\kappa = 0.5$  is truly free of crystalline order. The smooth evolution of  $\mathcal{O}$  with  $\kappa$  rules out the possibility of a first-order phase transition. Indeed, while the order parameter becomes exponentially small at small  $\kappa$  (numerical precision limits our study to  $\kappa \gtrsim 0.25$ ), a careful analysis of the numerical data indicates that, at any finite  $\kappa$ , it will not vanish as  $L \rightarrow \infty$ . Using standard methods of extrapolation [28], we find that the order parameter for  $L \rightarrow \infty$  is given by  $\exp(-1/\Gamma(\kappa)^2)$ , where  $\Gamma(\kappa)$  vanishes linearly at  $\kappa = 0$

[see inset in Fig. 1(a)]. According to this result, density wave order exists as long as  $\kappa \neq 0$ .

In terms of the quantum Hall connection, the fact that the Laughlin liquid on a torus with any finite  $L_y$  (and infinite  $L_x$ ) has a nonzero density wave order is very surprising (to us at least). From the perspective of a 1D lattice problem, however, it can be shown that, in general, there exists a local order parameter that can be used to distinguish the degenerate ground states required by Oshikawa's argument [29,30]. However, e.g., for  $\kappa = 0.25$ , the crystal order is 15 orders of magnitude weaker than that at  $\kappa = 1.5$ . Hence, for all practical purposes, this is a state without order.

In the following, we carefully study the energy gap as a function of  $\kappa$  to demonstrate that (i) the system is a Mott insulator for all  $\kappa$ , and (ii) there is no quantum phase transition between the large and small  $\kappa$  regimes. In Fig. 1(b), we show the numerically computed energy gap vs  $\kappa$  for system sizes up to  $L = 30$ . Only data for even particle numbers (to avoid even/odd effect) are used. The limit  $L \rightarrow \infty$  and  $\kappa$  fixed amounts to studying a Hall torus with  $L_x \rightarrow \infty$  and fixed  $L_y$  fixed. To avoid a finite size effect from small  $L_x$ , we restrict  $L_x > 12l_B$  or, equivalently,  $\kappa L > 12$  (for  $L = 30$ ,  $\kappa > 0.4$ ). With these restrictions, the data points collapse onto a single curve, as shown in Fig. 1(b). Note that, according to this result, the gap remains robust even in the "transition regime" around  $\kappa = 0.5$ . According to quantum Hall physics, the energy gap should not depend on  $L_x$  or  $L_y$  when both are much greater than  $l_B$ . Thus, for  $\kappa < 0.4$  (or  $L_y > 5\pi l_B$  for the Hall torus), we expect little variation in the extrapolated gap value. The asterisk in Fig. 1(b) is obtained by extrapolating the gap value at  $\kappa^*(L)$  to  $L = \infty$  [31].

From these results, the energy gap is nonzero for all  $\kappa$  [32]. However, in the inset in Fig. 1(b), we demonstrate that two different mechanisms cause the energy gap at small and large  $\kappa$ . For large  $\kappa$ , the gap is the energy penalty of having next-nearest-neighbor particles. The latter quantity is proportional to  $f(1)^2 \mathcal{O}$ . However, the energy gap plotted against  $f(1)^2 \mathcal{O}$  exhibits two branches, indicating a different mechanism for the energy gap at small  $\kappa$  values. Hence, the small  $\kappa$  state is not just an ordinary crystal with extremely small order. Thus, even in the most unfavorable case (1D), our approach has succeeded in producing a Mott insulator that is practically featureless. In higher dimensions, where there are no theorems forbidding featureless states, we expect Eq. (5) will lead to *truly* featureless Mott insulators (see later discussion on quantum dimer models).

Our analysis also leads to important implications for the fractional quantum Hall liquid on a torus. In the quantum Hall community, it is well known that, when  $L_y$  is large compared to  $l_B$  (i.e., when  $\kappa$  is small), the Laughlin state on the torus should be practically indistinguishable from that in the infinite 2D plane ( $L \rightarrow \infty$ ,  $\kappa = 0$ ). Quite unexpectedly, however, our findings imply that the Laughlin state in

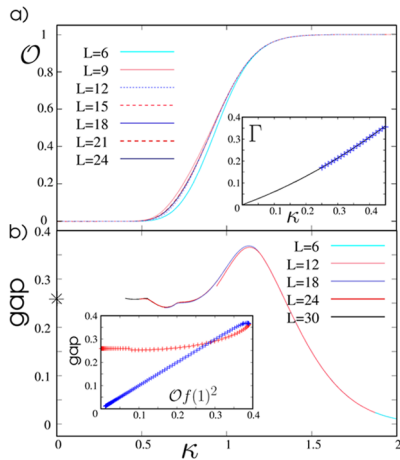


FIG. 1 (color online). The density wave order parameter and the energy gap of Eq. (6) as a function of  $\kappa$ . Data are obtained for  $L \leq 30$ . (a) The crystalline order parameter. Inset: The quantity  $\Gamma = (-\log \mathcal{O})^{-1/2}$ , extrapolated to  $L = \infty$  using the alternating epsilon algorithm [28]. The smooth curve is a polynomial fit of the data for  $0.25 < \kappa < 0.45$  (crosses). (b) The energy gap for data restricted to  $\kappa L > 12$  and even particle number (see text). Inset: The energy gap is plotted versus  $\mathcal{O}f(1)^2$ . Dark gray (blue) and light gray (red) symbols are for  $\kappa > 1.1$  and  $\kappa < 1.1$ , respectively. Two distinct branches are clearly visible.

this regime is adiabatically connected to a state with strong crystalline order at small  $L_y$  [33]. The above result reveals a danger in overinterpreting adiabatic continuity, because the Laughlin state is clearly qualitatively different from the electron crystal. However, since discrete quantum numbers are preserved by adiabatic continuity, both the ground state degeneracy and the quasiparticle charge are the same in the quantum Hall and the crystal states. The obvious threefold degeneracy of the electron crystal originates from the three distinct center-of-mass positions of the system. The excited states of the crystal are domain walls carrying  $\pm 1/3$  charge due to the Su-Schrieffer counting argument [34].

Our findings are consistent with the notion that the Laughlin state on a finite torus has topological order. As proposed by Wen and Niu [3], unlike in a symmetry broken state, disorder lifts the ground state degeneracy of a topologically ordered state by an amount that is exponentially small in the system size. In our case, such exponential dependence comes from the weak order parameter itself. Based on this, we conclude that, despite the common perception, the presence of symmetry breaking order should not be taken as excluding topological order.

Our belief that Eq. (5) can lead to truly featureless Mott insulators in higher dimensions is supported by the following connection to the quantum dimer model [10]. The quantum dimer model is a special case of the bosonic version of Eq. (5) in two dimensions. Indeed, once dimers are reinterpreted as point bosons residing at bond centers, the quantum dimer Hamiltonian becomes a center-of-mass position conserving model described by Eq. (5). Moessner and Sondhi [11] argued that the quantum dimer model on a triangular lattice exhibits an unordered quantum phase. This supports our belief that Eq. (5) leads to truly featureless Mott states in higher dimensions.

Finally, although we have focused on Mott insulators, gapless states described by Eq. (5) are equally interesting. We can prove that such a gapless system cannot be an ordinary metal (fermions) or a superfluid (bosons), because center-of-mass position conservation implies the absence of Drude weight/superfluid density. For example, in one space dimension, such a gapless liquid will not fall within the usual Luttinger liquid paradigm.

To conclude, we have presented the idea that simultaneous conservation of c.m. position and momentum can lead to fractionally filled featureless Mott insulators. It is remarkable that the simple model given by Eq. (5) unifies systems as diverse as the quantum Hall liquid and the quantum dimer liquid, whose effective field theories are as different as Chern-Simons gauge theory and Z2 gauge theory, respectively. We hope our results will spur the exploration of new directions and lead to a wealth of new states of matter that have, so far, escaped our attention.

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