Renormalization-group analysis of coupled superconducting order and stripe order in 1+1 dimensions

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In this paper, we perform a renormalization-group analysis in the $(1+1)$-dimensional version of a previously proposed effective-field theory describing (quantum) fluctuating stripe and superconductor orders. We find four possible phases corresponding to stripe order/disorder combined with superconducting order/disorder.

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I. INTRODUCTION

In $La_{2-x}Sr_xCuO_4$ compounds, there are three well-established ordering tendencies: antiferromagnetism, superconductivity, and charge/spin stripes.[1] Some experiments indicate that stripes and superconductivity can even coexist in these compounds.[2] Furthermore, neutron-scattering experiments by Lake *et al.*[3] show that a moderate magnetic field can have large effects on the incommensurate magnetic fluctuations. This is widely taken as evidence suggesting the stabilization of stripes by the magnetic field.

In mean-field theory, when two order parameters are in close competition it is possible for them to coexist in a certain region of the phase diagram.[4] In such a coexistence region, quantum fluctuations of both order parameters dominate the low-energy physics. In a recent paper, Lee[5] examined such a situation. The paper described how the Goldstone modes of stripe and superconducting orders and their respective topological defects interact.

We stress that the theory presented in Ref. 5 differs in important ways from the conventional self-dual charge-density wave/superconductivity action in one dimension. Indeed, in one dimension the displacement field of the charge-density wave is conjugate to the phase of the superconducting order. As a result the charge-density wave and the superconducting orders are mutually exclusive (i.e., whenever superconducting susceptibility strongly diverges, the charge-density wave susceptibility does not and vice versa). In contrast, in the theory of Ref. 5 there exists a generic region in the phase diagram where both orders exist.

In this paper, we examine in detail a one-dimensional analog of the model studied in Ref. 5. The motivation for this is that well-developed calculational methods (such as the renormalization-group) can be used to analyze the phase structure of the model. This can be used to check the correctness of the asserted phase structure in Ref. 5.

Now we describe the theory proposed by Lee.[5] Since the stripe order is a one-dimensional charge-density wave, its Goldstone mode (i.e., stripe displacement) is a $U(1)$ scalar.[5] The superconducting order, of course, also possesses a $U(1)$ Goldstone mode. The important question is: how do these two $U(1)$ modes couple together? A hint of how this coupling works comes from the experimental fact that the period of incommensurate spin correlation decreases as the doping density increases. Motivated by this, Lee[5] constructed the following Lagrangian density:

$$\mathcal{L}=\frac{1}{2K_{\phi\mu}}J_\mu^2+\frac{1}{2K_{\rho\mu}}q_\mu^2+J_\mu\overline{\phi_0}\partial_\mu\phi_0+q_x(\overline{\rho_0}\partial_x\rho_0-ig_1J_t)+q_t(\overline{\rho_0}\partial_t\rho_0-ig_2J_x). \tag{1}$$

In the above, $\phi_0=e^{i\theta_s}$ is the U(1) phase factor of the superconducting order parameter, and $\rho_0=e^{i\theta_p}$ is the phase factor of the stripe order. That is, $\theta_p=(2\pi/\lambda)\hat{\mathbf{x}}\cdot\mathbf{u}(x,t)$, with λ the stripe period and $\mathbf{u}(x,t)$ the displacement field of the stripe order. J_μ and q_μ are auxiliary fields coupling to the superconducting and stripe phases, respectively. These auxiliary fields have the physical interpretation of energy-momentum currents. (In this paper, greek indices run over x,t, and repeated indices are summed.)

Without the coupling $(g_{1,2}=0)$, integrating out J_μ and q_μ produces the field theory for two independent U(1) Goldstone modes and their respective vortices. The effect of the coupling is to favor stripe displacement $\mathbf{u}$ in the presence of local charge imbalance J_t.

To analyze Eq. (1), Lee used a duality transformation plus an educated guess about the four possible quantum phases corresponding to combinations of stripe and superconducting order/disorder. In this paper, we study a one-dimensional version of Eq. (1), applying the well-developed techniques of duality transformation and the renormalization-group to determine the possible phases in a more unbiased fashion. We find that all four combinations of stripe order/disorder and superconducting order/disorder are stable phases. This supports the conjectured phase structure in Ref. 5.

In the following, we use a real-space renormalization procedure similar to that used by Kosterlitz and Thouless to treat the phase transition of the two-dimensional Coulomb gas.[6,7] In Sec. II, we derive the vortex action (the vortex of the stripe order parameter is the dislocation). In Sec. III we obtain the renormalization-group recursion relations for the coupling constants in that theory. As in the Kosterlitz-Thouless theory, we make the small vortex fugacity approximation. We analyze the implications of these flows for phase stability in Sec. IV.

II. DUALITY TRANSFORMATION TO TWO-SPECIES COULOMB GAS

Following the work of Jose *et al.*[7] we first perform a duality transformation and write the theory in terms of vortex degrees of freedom.

Starting with Eq. (1), we first separate the phase of ϕ_0 and ρ_0 into a topologically trivial part and a topologically nontrivial part:

$$\phi_0 = e^{i\eta_0}\phi,$$

$$\rho_0 = e^{i\xi_0}\rho. \tag{2}$$

In the above, η_0 and ξ_0 are single valued, while ϕ and ρ contain configurations with nonzero windings. After integrating over the topologically trivial phases (η_0,ξ_0), we obtain two conservation laws,

$$\partial_\mu J_\mu = 0,$$

$$\partial_\mu q_\mu = 0. \tag{3}$$

To explicitly fulfill these conservation laws, we write $J_\mu = \epsilon_{\mu\nu}\partial_\nu\Lambda$ and $q_\mu = \epsilon_{\mu\nu}\partial_\nu\chi$, where χ and Λ are scalar fields. Substitution leads to

$$\mathcal{L} = \frac{1}{2K_{\rho\bar{\mu}}}(\partial_\mu\chi)^2 + \frac{1}{2K_{\phi\bar{\mu}}}(\partial_\mu\Lambda)^2 + \epsilon_{\mu\nu}\partial_\nu\Lambda\overline{\phi}\partial_\mu\phi + \epsilon_{\mu\nu}\partial_\nu\chi\overline{\rho}\partial_\mu\rho + ig_1\partial_t\chi\partial_x\Lambda + ig_2\partial_x\chi\partial_t\Lambda. \tag{4}$$

In the above the index $\bar{\mu}$ denotes x if μ is t and vice versa. Upon integrating by parts and identifying the vortex densities $N = i\epsilon_{\mu\nu}\partial_\nu(\overline{\rho}\partial_\mu\rho)$ and $M = i\epsilon_{\mu\nu}\partial_\nu(\overline{\phi}\partial_\mu\phi)$, the Lagrangian density becomes

$$\mathcal{L} = \frac{1}{2K_{\rho\bar{\mu}}}(\partial_\mu\chi)^2 + \frac{1}{2K_{\phi\bar{\mu}}}(\partial_\mu\Lambda)^2 + i\Lambda M + i\chi N - i(g_1+g_2)\Lambda\partial_t\partial_x\chi. \tag{5}$$

The above equation can be written in momentum space as

$$\mathcal{L} = \frac{1}{2}(\chi(\mathbf{k}) \quad \Lambda(\mathbf{k}))^* \begin{pmatrix} \dfrac{k_\mu^2}{K_{\rho\bar{\mu}}} & iGk_xk_t \\ iGk_xk_t & \dfrac{k_\mu^2}{K_{\phi\bar{\mu}}} \end{pmatrix} \begin{pmatrix} \chi(\mathbf{k}) \\ \Lambda(\mathbf{k}) \end{pmatrix} + i(\chi(\mathbf{k}) \quad \Lambda(\mathbf{k}))^* \begin{pmatrix} N(\mathbf{k}) \\ M(\mathbf{k}) \end{pmatrix}, \tag{6}$$

where $G = g_1 + g_2$. Integrating out the χ and Λ fields then produces

$$\mathcal{L} = \frac{1}{2}(N \quad M)^* \frac{1}{\det} \begin{pmatrix} \dfrac{k_\mu^2}{K_{\phi\bar{\mu}}} & -iGk_xk_t \\ -iGk_xk_t & \dfrac{k_\mu^2}{K_{\rho\bar{\mu}}} \end{pmatrix} \begin{pmatrix} N \\ M \end{pmatrix}, \tag{7}$$

$$\det = \left(\frac{k_\mu^2}{K_{\rho\bar{\mu}}}\right)\left(\frac{k_\nu^2}{K_{\phi\bar{\nu}}}\right) + G^2k_x^2k_t^2. \tag{8}$$

Equation (8) is the starting point of our renormalization-group analysis. It describes a system of two interacting (anisotropic) Coulomb gases—the vortices of the superconducting order parameter and the dislocations of the stripe order parameter. Inspired by the work of Kosterlitz and Thouless, we perform a real-space renormalization-group analysis of Eq. (8) in the following.

III. RENORMALIZATION GROUP ANALYSIS

Equation (8) is more complicated than the one species Coulomb gas problem in two respects: (1) there are two species of vortices and (2) the interactions are not rotationally invariant (i.e., the interaction depends not only on the distance between vortices but also on their relative orientation). In order to complete the renormalization-group program we have to characterize the interaction in terms of a discrete set of coupling constants. One way of achieving this is to Fourier transform the angular dependence of the vortex-vortex interaction. In momentum space each element of the interaction matrix is of the form $G(\mathbf{k}) = G(k,\theta) = g(\theta)/k^2$ (θ is the angle made by $\mathbf{k}$ and the k_x axis). Therefore, we expand each of these terms in a Fourier series, e.g., $g(\theta) = \Sigma_n a_n e^{(in\theta)}$. When transformed back into real space, our action then becomes

$$\mathcal{S} = \frac{1}{2}\int d^2\mathbf{R}_1 d^2\mathbf{R}_2 N(\mathbf{R}_1)G_N(\mathbf{R}_1-\mathbf{R}_2)N(\mathbf{R}_2) + M(\mathbf{R}_1)G_M(\mathbf{R}_1-\mathbf{R}_2)M(\mathbf{R}_2) + 2M(\mathbf{R}_1)i\Gamma(\mathbf{R}_1-\mathbf{R}_2)N(\mathbf{R}_2), \tag{9}$$

where

$$G_N(\mathbf{R}) = \int \frac{d^2\mathbf{k}}{(2\pi)^2}\left(\sum_{n=-\infty}^{\infty} a_n e^{in\theta}\right)\frac{e^{i\mathbf{k}\cdot\mathbf{R}}}{k^2}, \tag{10}$$

$$G_M(\mathbf{R}) = \int \frac{d^2\mathbf{k}}{(2\pi)^2}\left(\sum_{n=-\infty}^{\infty} \alpha_n e^{in\theta}\right)\frac{e^{i\mathbf{k}\cdot\mathbf{R}}}{k^2}, \tag{11}$$

$$\Gamma(\mathbf{R}) = \int \frac{d^2\mathbf{k}}{(2\pi)^2}\left(\sum_{n=-\infty}^{\infty} c_n e^{in\theta}\right)\frac{e^{i\mathbf{k}\cdot\mathbf{R}}}{k^2}. \tag{12}$$

In the above

$$a_n = (-1)^n a_{-n}^*,$$

$$\alpha_n = (-1)^n \alpha_{-n}^*,$$

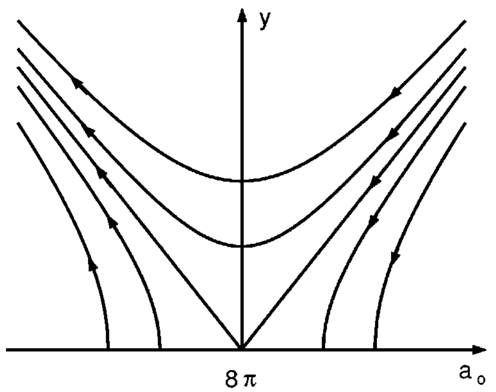

FIG. 1. Renormalization-group flow for the Kosterlitz-Thouless transition. The fixed points are at the $y=0$ axis. In our model this corresponds to only y, a_0 nonzero.

$$c_n = (-1)^n c^*_{-n} \tag{13}$$

to ensure that the interaction functions are real. We stress that because of the angular dependence of Eqs. (10)–(12), G_N, G_M, and Γ depend not only on the distance $|\mathbf{R_1}-\mathbf{R_2}|$ but also on the relative orientation $(\mathbf{R_1}-\mathbf{R_2})/|\mathbf{R_1}-\mathbf{R_2}|$. In general, a_n and α_n are nonzero only for even n. The physical reason for this is indistinguishability of two charges of the same type (for details, see the Appendix). c_n can be nonzero for both odd and even n.

The limit where all the a_n, α_n, and c_n are zero except a_0 and α_0 describes two decoupled isotropic two-dimensional *X-Y* models in their Coulomb gas representations—the Kosterlitz and Thouless problem. Before we attack Eq. (8), as a warm up, let us briefly review the Kosterlitz-Thouless results for the one-component system. In the renormalization-group approach one integrates out one pair of tightly bound dipole (i.e., a dipole with $r_c+dr_c<$ size $<r_c$) at a time. The renormalization-group proceeds iteratively by treating r_c as a running length scale. The two coupling constants in this case are the vortex-vortex interaction strength a_0 and vortex fugacity $y=e^{-\mu}$, where μ is the core energy of vortices. In the limit of $y\ll 1$, the renormalization-group equations for a_0 and y are given by

$$\frac{dy}{dl} = y\left(2-\frac{a_0}{4\pi}\right), \tag{14}$$

$$\frac{da_0}{dl} = -\pi y^2 a_0^2. \tag{15}$$

The above equations have the entire $y=0$ axis as fixed points. However, depending on whether $a_0-8\pi$ is positive/negative, the fixed point is stable/unstable. The point $y=0, a_0=8\pi$ is a critical point. Near it, the flow trajectories are given by

$$a_0^2 - (4\pi)^4 y^2 = C. \tag{16}$$

Here C is a constant labeling each trajectory. This flow is shown in Fig. 1. Note that the $C=0$ separatrix $y=[1/(4\pi)^4](a_0-8\pi)$ separates the basins of attraction for the ordered and disordered phases. In the ordered phase the density of vortices renormalizes to zero $(y\to 0)$ at large length scales, signifying the presence of a bound dipole phase. In the disordered phase the density of vortices increases (y increases) at large length scales, signifying the existence of a vortex plasma (unbound dipole) phase.

The two-component vortex gas problem we are facing is not so different. However, we have to keep track of all the Fourier coefficients in Eqs. (10)–(12) and examine how they renormalize. Interestingly, even in the presence of these anisotropic interactions, the Kosterlitz-Thouless renormalization-group program closes.

All the technical details are given in the Appendix. Here we just note the following point. Since the "Coulomb charge" of the superconducting vortices is not related to the Coulomb charge of the stripe vortices, only intraspecies dipoles are possible. This implies that the positions of vortices belonging to different species do not have to obey the constraint that the minimum distance is r_c. To lowest order in y the resulting renormalization-group equations are given by

$$\frac{dy_N}{dl} = y_N\left(2-\frac{a_0}{4\pi}\right),$$

$$\frac{dy_M}{dl} = y_M\left(2-\frac{\alpha_0}{4\pi}\right),$$

$$\frac{da_n}{dl} = -\pi\sum_{k,m}\delta_{n,k+m}[y_N^2 a_k a_m + y_M^2(-1)^{m+1}c_k c_m],$$

$$\frac{d\alpha_n}{dl} = -\pi\sum_{k,m}\delta_{n,k+m}[y_M^2 \alpha_k \alpha_m + y_N^2(-1)^{m+1}c_k c_m],$$

$$\frac{dc_n}{dl} = -\pi\sum_{k,m}\delta_{n,k+m}(y_N^2 a_k + y_M^2 \alpha_k)c_m. \tag{17}$$

Note that the renormalization of $y_{N,M}$ only depends on the isotropic part of the coupling (a_0 and α_0). The renormalization of an anisotropy coefficient, (e.g., a_n) includes many terms. Each term is a quadratic function of a_k, α_k, or c_k. If we set all coupling constants except a_0 and α_0 to zero we recover the Kosterlitz-Thouless flow equations (for two separate species). It is easily verified that the condition for G_N, G_M, and Γ to be real $[a_n=(-1)^n a^*_{-n}$, etc.] is preserved by these flow equations. It is also clear from the form of these equations that these coefficients form a closed set under renormalization.

IV. PHASES OF THE TWO-SPECIES COULOMB GAS

Equation (17) predicts fixed points for $y_M=y_N=0$ and a_n, α_n, c_n can be anything. As in the normal Kosterlitz-Thouless case, we interpret $y=0$ as the absence of unbound dipoles. From the first line of Eq. (17), $y_N=0$ is linearly stable when $a_0>8\pi$. For $a_0>8\pi$, the renormalization-group brings y_N to larger values. Similar statements hold for α_0 and y_M. This suggests the presence of four phases depending on whether the vortices of N or M species form dipoles or unbind.

However, this is not quite enough for our purposes. What we really need to know is whether all four phases can be reached by varying the five parameters in Eq. (7). Put in another way, the physical system of Eq. (7) is in a five dimensional subspace of the infinite-dimensional space formed by the a_n, α_n, and c_n. We need to check which phases can be reached by trajectories originating in the physical subspace, not just which phases exist for the infinite dimensional space.

In order to obtain a tractable problem, in the following we concentrate on the case in which $K_{\rho\mu}=K_\rho$, $K_{\phi\mu}=K_\phi$ and $GK_{\rho,\phi}\ll 1$. When $GK_{\rho,\phi}$ are small, it is easy to evaluate the Fourier coefficients in Eqs. (10)–(12) in powers of $GK_{\rho,\phi}$. If $GK_{\rho,\phi}=O(\epsilon)$, we find that the leading contribution to a_n, α_n, c_n is $O(\epsilon^{|n|/2})$. For the specific form of interaction in Eq. (7) it is simple to see that besides Eqs. (13) there are additional constraints on a_n, α_n, and c_n:

$$a_n=\alpha_n=0 \quad \text{unless} \quad n=4m,$$

$$c_n=0 \quad \text{unless} \quad n=4m+2. \tag{18}$$

All of these conditions are preserved by the flow equations. In terms of the original parameters in Eq. (7), the nonvanishing coefficients up to order ϵ are

$$a_0=K_\rho(1-K_\phi K_\rho G^2/8),$$

$$\alpha_0=K_\phi(1-K_\phi K_\rho G^2/8),$$

$$c_2=c^*_{-2}=iGK_\rho K_\phi/4. \tag{19}$$

In the following, we truncate the space considered to only these coefficients, which is correct to lowest order in ϵ. Furthermore, this restricts us to a five-dimensional space of parameters, which we can take to be independently determined by the five parameters in Eq. (7).

In this case, the flow equations (17) become

$$\frac{dy_N}{dl}=y_N\left(2-\frac{a_0}{4\pi}\right), \tag{20}$$

$$\frac{dy_M}{dl}=y_M\left(2-\frac{\alpha_0}{4\pi}\right), \tag{21}$$

$$\frac{da_0}{dl}=-\pi(y_N^2a_0^2-y_M^2|c_2|^2), \tag{22}$$

$$\frac{d\alpha_0}{dl}=-\pi(y_M^2\alpha_0^2-y_N^2|c_2|^2), \tag{23}$$

$$\frac{dc_2}{dl}=-\pi(y_N^2a_0c_2+y_M^2\alpha_0c_2). \tag{24}$$

First, by multiplying Eq. (24) by c_2^* and then adding the result to its complex conjugate, we obtain

$$\frac{d|c_2|^2}{dl}=-2\pi(y_N^2a_0+y_M^2\alpha_0)|c_2|^2. \tag{25}$$

Using this in Eqs. (22) and (23) then gives us

$$\frac{da_0}{dl}=-\pi y_N^2(a_0^2+|c_2|^2)-\frac{1}{16\pi}\frac{d|c_2|^2}{dl}, \tag{26}$$

$$\frac{d\alpha_0}{dl}=-\pi y_M^2(\alpha_0^2-|c_2|^2)-\frac{1}{16\pi}\frac{d|c_2|^2}{dl}. \tag{27}$$

At this point, we examine closely the region of parameter space around the critical point by making the change of variables

$$a=a_0-8\pi,$$

$$\alpha=\alpha_0-8\pi,$$

$$c=c_2-\overline{c}.$$

In the above, $\overline{c}$ is the fixed point of c_2. After some algebra, and keeping terms to lowest order in a, α, c, y_N, and y_M, the flow equations for a and α are

$$\frac{dy_N^2}{dl}=\frac{1}{2\pi}y_N^2a, \tag{28}$$

$$\frac{dy_M^2}{dl}=\frac{1}{2\pi}y_M^2\alpha, \tag{29}$$

$$\frac{da^2}{dl}=-2\pi y_N^2a[(8\pi)^2+|\overline{c}|^2]-\frac{a}{16\pi}\frac{d|c_2|^2}{dl}, \tag{30}$$

$$\frac{d\alpha^2}{dl}=-2\pi y_M^2\alpha[(8\pi)^2-|\overline{c}|^2]-\frac{\alpha}{16\pi}\frac{d|c_2|^2}{dl}. \tag{31}$$

Finally, we can combine these equations to yield

$$\frac{d(a^2-Xy_N^2)}{dl}=\frac{dC_N}{dl}=-\frac{a}{8\pi}\frac{d|c_2|^2}{dl}, \tag{32}$$

$$\frac{d(\alpha^2-Xy_M^2)}{dl}=\frac{dC_M}{dl}=-\frac{\alpha}{8\pi}\frac{d|c_2|^2}{dl}. \tag{33}$$

Here, $X=(4\pi)^4(1+|\overline{c}|^2/(8\pi)^2)$. To understand these equations, note that the quantity in parentheses on the left-hand side of each equation is precisely of the form of the contour numbers for trajectories in the Kosterlitz-Thouless case [cf. Eq. (16)], with the slope of the separatrix renormalized from $[1/(4\pi)^4]$ to $1/X$. In the absence of interactions ($c=0$), these contour numbers C_N and C_M are conserved, but in the presence of interactions, the renormalization-group pushes the flow from one contour to the next. Furthermore, by Eqs. (32) and (25), for $a>0$ (i.e., $a_0>8\pi$) the contour number C_N increases, while for $a<0$ (i.e., $a_0<8\pi$) C_N decreases. Similarly for C_M and α. The resulting flow is diagrammed in Fig. 2.

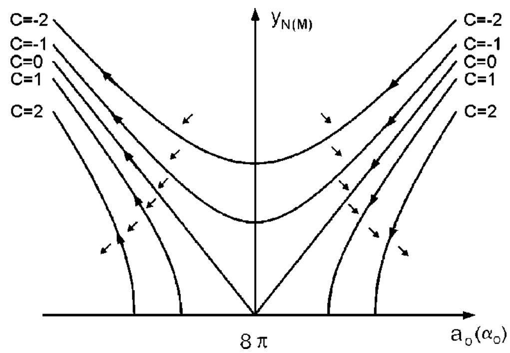

FIG. 2. Renormalization-group flow for weakly coupled superconducting(stripe) vortex gases, with contour numbers C labeled. The fixed points are at the $y=0$ axis. With nonzero coupling, the flow moves from one contour to the next as indicated by the arrows. It is clear that even in the presence of coupling for each gas, there are still two phases corresponding to $y\to 0$ and y increasing.

It is clear from this that for a trajectory originating below the separatrices $[y_N<(a_0-8\pi)/X$ and $y_M<(\alpha_0-8\pi)/X]$, the flow leads to both y_N and y_M zero. In other words, there is a stable phase with both types of vortices bound as dipoles, corresponding to a stripe ordered/superconducting ordered phase. If the trajectory starts with $(a_0-8\pi)<0$ and $(\alpha_0-8\pi)<0$, the flow leads to both y_N and y_M increasing and unbound dipoles of both species. Thus, there is a stripe disordered/superconducting disordered phase. Finally, in the mixed case, e.g., $y_N<(a_0-8\pi)/X$ and $(\alpha_0-8\pi)<0$, y_N flows to zero but y_M increases. Thus there are phases with stripe disorder/superconducting order and stripe order/superconducting disorder.

We have seen that with weak (g_1+g_2) all four phases corresponding to stripe order/superconducting order, stripe disorder/superconducting order, stripe order/superconducting disorder, and stripe disorder/superconducting disorder are stable and can be realized in the system described by Lee's theory in $(1+1)$ dimensions [Eq. (1)]. We emphasize that we have analyzed only the case with isotropic couplings $K_{\rho\mu}=K_\rho$ and $K_{\phi\mu}=K_\phi$ and weak coupling. In this case, the phase diagram in the K_ϕ/K_ρ plane is shown in Fig. 3.

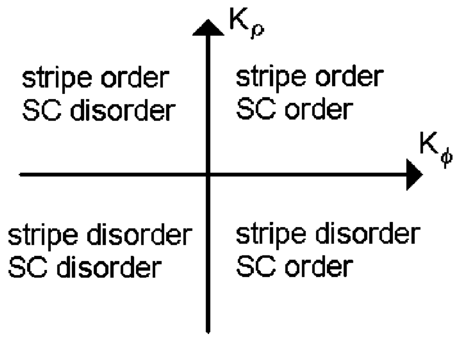

FIG. 3. Phase diagram for interacting stripe and superconducting order, for isotropic couplings K_ϕ and K_ρ and weak interaction G. Only the K_ϕ-K_ρ plane is shown. Although different values of G correspond to different long-range interactions, G does not control the existence of stripe/superconducting order or disorder.

V. SUMMARY

The main results of this paper are the interacting Coulomb gas representation of the competing stripe and superconducting orders, Eq. (8); and the renormalization-group flow, Eq. (17). Analysis of these flow equations shows that the $(1+1)$-dimensional version of the theory proposed in Ref. 5 supports stable phases corresponding to stripe order/superconducting order, stripe disorder/superconducting order, stripe order/superconducting disorder, and stripe disorder/superconducting disorder.

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APPENDIX: DETAILS OF RENORMALIZATION-GROUP CALCULATION

Here we present the details for the renormalization of the system described by Eqs. (7) and (8), and parametrized in Fourier coefficients via Eqs. (10)–(12). The details closely follow the procedure used by Jose *et al.*[7] The basic idea is to introduce a small length scale cutoff r_c, and then integrate out configurations with pairs of the same type of charge that are r_c+dr_c apart to find a new system with a longer minimum length scale.

In our action, Eq. (9), the fields N and M consist of point charges, e.g., $N(\mathbf{r})=\Sigma_\alpha N_\alpha \delta(\mathbf{r}-\mathbf{r}_\alpha)$. We will express the terms, Eq. (9), involving G_N and G_M as sums over pairs of these charges. This results in the cancellation of all odd fourier components of G_N and G_M, i.e., a_n and α_n are zero for odd n. To see this, note that in reexpressing the sum as a sum over pairs, we combine terms like $N_\alpha N_\beta G_N(\mathbf{R}_\alpha-\mathbf{R}_\beta)+N_\beta N_\alpha G_N(\mathbf{R}_\beta-\mathbf{R}_\alpha)$. The two G_N's above differ by reversing the relative vector (i.e., $\theta\to\theta+\pi$). Since odd Fourier components pick up a relative minus sign under this reversal, they cancel each other, and only the even Fourier components survive. On the other hand, since the terms with Γ describe the interaction between distinguishable vortices, we cannot convert them into sums over pairs, and therefore the Fourier coefficients c_n can be nonzero for both odd and even n.

As in the Kosterlitz-Thouless case, G_N and G_M diverge logarithmically at short length scales. In Eq. (7) this translates to divergence when $\mathbf{R}_1=\mathbf{R}_2$. To remove this divergence, we must enforce charge neutrality $(\Sigma_\alpha N_\alpha=\Sigma_\alpha M_\alpha=0)$ and impose a small distance cutoff r_c.[8] To account for the microscopic physics lost in this procedure, we introduce core energies Δ_N and Δ_M for the vortices (charges). After this our action is written as

$$\mathcal{S}=\sum_{(\alpha,\beta)} N_\alpha N_\beta G_N(\mathbf{R}_\alpha-\mathbf{R}_\beta)+\sum_{(\alpha,\beta)} M_\alpha M_\beta G_M(\mathbf{R}_\alpha-\mathbf{R}_\beta)$$
$$+\sum_{\alpha,\beta} M_\alpha N_\beta i\Gamma_N(\mathbf{R}_\alpha-\mathbf{R}_\beta)+\sum_\alpha N_\alpha^2\Delta_N+\sum_\alpha M_\alpha^2\Delta_M, \tag{A1}$$

where (α,β) denotes a sum over pairs and α,β denotes an unrestricted sum over both α and β. At this point we make the simplifying assumption that Δ_N and Δ_M are very large, so we may restrict to $N_\alpha, M_\alpha=\pm 1$. Introducing the fugacities $y_N=e^{-\Delta_N}$ and $y_M=e^{-\Delta_M}$, we can write the partition function as a sum over j N dipoles and k M dipoles as

$$Z=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} y_N^{2j}y_M^{2k}\frac{1}{j!^2}\frac{1}{k!^2}$$
$$\times\int \frac{d^2\mathbf{x}_1}{r_c^2}\cdots\frac{d^2\mathbf{x}_{2j}}{r_c^2}\int\frac{d^2\mathbf{z}_1}{r_c^2}\cdots\frac{d^2\mathbf{z}_{2k}}{r_c^2}e^{-\tilde{S}}, \tag{A2}$$

$$\tilde{S}=\sum_{(\alpha,\beta)} N_\alpha N_\beta G_N(\mathbf{x}_{\boldsymbol{\alpha}}-\mathbf{x}_\beta)+\sum_{(\alpha,\beta)} M_\alpha M_\beta G_M(\mathbf{z}_\alpha-\mathbf{z}_\beta)$$
$$+\sum_{\alpha,\beta} M_\alpha N_\beta i\Gamma_N(\mathbf{z}_\alpha-\mathbf{x}_\beta). \tag{A3}$$

The first step in the real-space renormalization procedure is to integrate over dipoles of size r_c+dr_c. Since $y_N, y_M \ll 1$, we need only consider configurations with one dipole of either type. Furthermore, mixed M-N dipoles are not integrated out because doing so would violate overall charge neutrality for each species. This means that the M and N charges are configured freely with respect to each other. Writing out these single-dipole contributions, the partition function gains an extra factor:

$$e^{-\tilde{s}}\to e^{-\tilde{s}}\left\{1+\int\frac{d^2\mathbf{R}_N}{r_c^2}\int_{r_c}^{r_c+dr_c}\frac{d^2\mathbf{r}_N}{r_c^2}y_N^2e^{-S'_N}\right.$$
$$\left.+\int\frac{d^2\mathbf{R}_M}{r_c^2}\int_{r_c}^{r_c+dr_c}\frac{d^2\mathbf{r}_M}{r_c^2}y_M^2e^{-S'_M}+O(y^4)\right\}, \tag{A4}$$

$$S'_N=-G_N(\mathbf{r}_N)$$
$$+\sum_\alpha^{2k} N_\alpha\left(G_N\left(\mathbf{R}_N+\frac{\mathbf{r}_N}{2}-\mathbf{x}_\alpha\right)-G_N\left(\mathbf{R}_N-\frac{\mathbf{r}_N}{2}-\mathbf{x}_\alpha\right)\right)$$
$$+\sum_\alpha^{2j} M_\alpha\left(i\Gamma\left(\mathbf{R}_N+\frac{\mathbf{r}_N}{2}-\mathbf{z}_\alpha\right)-i\Gamma\left(\mathbf{R}_N-\frac{\mathbf{r}_N}{2}-\mathbf{z}_\alpha\right)\right), \tag{A5}$$

$$S'_M=-G_M(\mathbf{r}_M)$$
$$+\sum_\alpha^{2j} M_\alpha\left(G_M\left(\mathbf{R}_M+\frac{\mathbf{r}_M}{2}-\mathbf{z}_\alpha\right)-G_M\left(\mathbf{R}_M-\frac{\mathbf{r}_M}{2}-\mathbf{z}_\alpha\right)\right)$$
$$+\sum_\alpha^{2k} N_\alpha\left(i\Gamma\left(\mathbf{R}_M+\frac{\mathbf{r}_M}{2}-\mathbf{x}_\alpha\right)-i\Gamma\left(\mathbf{R}_N-\frac{\mathbf{r}_N}{2}-\mathbf{x}_\alpha\right)\right), \tag{A6}$$

where $\mathbf{R}$ signifies the center and $\mathbf{r}$ the separation of the dipole being integrated out. To obtain the above, we have used the fact that the combinatorial factor for $j+1$ pairs of charges, one of which is a dipole [i.e., $(j+1)^2/(j+1)!^2$], is equal to the combinatorial factor of j pairs of charges (i.e., $1/j!^2$).

To proceed, rewrite Eqs. (A5) and (A6) using identities like $(G_N[\mathbf{R}_N+(\mathbf{r}_N/2)-\mathbf{x}_\alpha]-G_N[\mathbf{R}_N-(\mathbf{r}_N/2)-\mathbf{x}_\alpha]) =\mathbf{r}_N\cdot\boldsymbol{\nabla}_{\mathbf{R}_N}G_N(\mathbf{R}_N-\mathbf{x}_\alpha)$. In Eq. (A4), these expressions enter into exponentials, which we expand. The linear terms in this expansion produce nothing interesting. To see this, note that there are two types of these terms: those with a gradient, and those without. The linear terms involving the gradient do not contribute since their integral with respect to $\mathbf{r}$ is zero. The linear terms involving no gradient $[-G_N(\mathbf{r}_N)$ and $-G_M(\mathbf{r}_M)]$ integrate to constants independent of j, k, and the N_α, M_α, so they merely produce an overall constant multiplying the partition function.

Since the linear terms in the expansion of the exponentials of Eq. (A4) are uninteresting, we look at the second-order terms. There are three types of terms here: those involving no gradients (e.g., G_NG_N); those involving one gradient (e.g., $G_N\mathbf{r}\cdot\nabla G_N$); and those involving two gradients (e.g., $\mathbf{r}\cdot\nabla G_N\mathbf{r}\cdot\nabla G_N$). The first type of term, with no gradients, integrates to a constant independent of j, k, and the N_α, M_α, so we ignore it.

The second type of term, with only one gradient, integrates to zero. For example, one such expression is $\Sigma_\alpha N_\alpha G(\mathbf{r}_N)\mathbf{r}_N\cdot\boldsymbol{\nabla}_{\mathbf{R}_N}G_N(\mathbf{R}_N-\mathbf{x}_{\boldsymbol{\alpha}})$. However, in each term of the sum, we can change integration variables to $\mathbf{R}'_N=\mathbf{R}_N-\mathbf{x}_{\boldsymbol{\alpha}}$ producing $\int\Sigma_\alpha N_\alpha G(\mathbf{r}_N)\mathbf{r}_N\cdot\nabla_{\mathbf{R}'_N}G_N(\mathbf{R}'_N)$. Charge neutrality then makes the sum over N_α zero.

The third type of term, with two gradients, does contribute to renormalization. A typical term of this type is (using the Fourier expansion of Γ)

$$\frac{1}{2!}\int\frac{d^2\mathbf{R}_N}{r_c^2}\int_{r_c}^{r_c+dr_c}\frac{d^2\mathbf{r}_N}{r_c^2}y_N^2\sum_{\alpha,\beta}M_\alpha M_\beta$$
$$\times\mathbf{r}_N\cdot\boldsymbol{\nabla}_{\mathbf{R}_N}\left[i\int\frac{d^2\mathbf{k}}{(2\pi)^2}\frac{1}{k^2}\left(\sum c_ne^{in\theta}\right)e^{i\mathbf{k}\cdot(\mathbf{z}_\alpha-\mathbf{R}_N)}\right]$$
$$\times\mathbf{r}_N\cdot\boldsymbol{\nabla}_{\mathbf{R}_N}\left[i\int\frac{d^2\mathbf{k}}{(2\pi)^2}\frac{1}{k^2}\left(\sum c_me^{im\theta}\right)e^{i\mathbf{k}\cdot(\mathbf{z}_\beta-\mathbf{R}_N)}\right].$$

Using $\mathbf{r}_N\approx r_c(\cos\phi,\sin\phi)$ we can integrate over $\mathbf{r}_N$ to obtain

$$-\frac{1}{2!}\frac{2\pi}{2}\frac{r_c^3 dr_c}{r_c^2}y_N^2\int\frac{d^2\mathbf{R}_N}{r_c^2}\sum_{\alpha,\beta}M_\alpha M_\beta\int\frac{d^2\mathbf{k}'}{(2\pi)^2}\frac{d^2\mathbf{k}}{(2\pi)^2}(-i\mathbf{k})\cdot(-i\mathbf{k}')\frac{1}{k^2k'^2}\sum_{n,m}c_nc_me^{in\theta+im\theta'}e^{i\mathbf{k}\cdot\mathbf{z}_\alpha+i\mathbf{k}'\cdot\mathbf{z}_\beta}e^{-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{R}_N}$$

$$=-\frac{\pi}{2!}\frac{dr_c}{r_c}y_N^2\sum_{\alpha,\beta}M_\alpha M_\beta\int\frac{d^2\mathbf{k}}{(2\pi)^2}\frac{1}{k^2}e^{i\mathbf{k}\cdot(\mathbf{z}_\alpha-\mathbf{z}_\beta)}\sum_{n,m}c_nc_me^{i(n+m)\theta}e^{im\pi}$$

$$=\frac{\pi}{2}\frac{dr_c}{r_c}y_N^2\sum_{\alpha,\beta}M_\alpha M_\beta\int\frac{d^2\mathbf{k}}{(2\pi)^2}\frac{1}{k^2}e^{i\mathbf{k}\cdot(\mathbf{z}_\alpha-\mathbf{z}_\beta)}\sum_n\left(\sum_{l,m}\delta_{n,l+m}c_lc_m(-1)^{m+1}\right)e^{in\theta}.$$

After many calculations like this and some algebra, we obtain a new partition function in the form of Eq. (A2) again, except that the interaction functions have been redefined to G_N', G_M', and Γ' via the new coefficients

$$a_n'=a_n-\pi\sum_{k,m}\delta_{n,k+m}(y_N^2a_ka_m+y_M^2(-1)^{m+1}c_kc_m)dl,$$

$$\alpha_n'=\alpha_n-\pi\sum_{k,m}\delta_{n,k+m}(y_M^2\alpha_k\alpha_m+y_N^2(-1)^{m+1}c_kc_m)dl,$$

$$c_n'=c_n-\pi\sum_{k,m}\delta_{n,k+m}(y_N^2a_k+y_M^2\alpha_k)c_mdl. \tag{A7}$$

Here $dl=dr_c/r_c$.

At this point, it is useful to examine more closely the structure of the contributions to the interaction functions and understand what sorts of interactions we are dealing with. To this end, we evaluate a typical term

$$\int\frac{d^2\mathbf{k}}{(2\pi)^2}e^{in\theta}\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{k^2}$$

and using $\mathbf{k}\cdot\mathbf{r}=kr\cos\phi$ this becomes

$$\frac{1}{(2\pi)^2}\int_0^{2\pi}d\theta\int_{1/L}^{\infty}\frac{dk}{k}e^{ikr\cos(\theta-\phi)}e^{in\theta}, \tag{A8}$$

$$=\frac{i^ne^{in\phi}}{2\pi}\int_{r/l}^{\infty}\frac{dx}{x}J_{-n}(x), \tag{A9}$$

where we have introduced the Bessel function J_n. Due to the oscillatory nature of the Bessel function, the integral is dominated by the infrared so we employ the asymptotic form $J_n(x)\sim(1/n!)(x/2)^n$ to obtain

$$\frac{(\pm i)^n}{2\pi n!}e^{in\phi}\left\{\text{const}+\int_{r/L}^{1}dx\frac{1}{x}\left(\frac{x}{2}\right)^{|n|}\right\}, \tag{A10}$$

where the negative sign refers to $n<0$. For the case $n\neq0$, the infrared part converges and we are left with a function of the spatial angle ϕ independent of r. For the case $n=0$, we obtain a logarithmic divergence in r. Using the cutoff r_c, the precise form for the $n=0$ contribution to the interaction functions is

$$\int\frac{d^2\mathbf{k}}{(2\pi)^2}\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{k^2}=-\frac{1}{2\pi}\ln\left|\frac{r}{r_c}\right|. \tag{A11}$$

Thus, in the interaction terms only the zero-mode Fourier component has any mention of r_c. To complete the renormalization-group program, we must write the partition function in a form containing only the renormalized core size $r_c'=(1+dl)r_c$. After we do this, the partition function is (up to an overall constant)

$$\begin{aligned}Z=&\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}y_N^{2j}y_M^{2k}(1+dl)^{4k+4j}\frac{1}{j!^2}\frac{1}{k!^2}\int\frac{d^2\mathbf{x}_1}{r_c'^2}\cdots\frac{d^2\mathbf{x}_{2j}}{r_c'^2}\\&\times\int\frac{d^2\mathbf{z}_1}{r_c'^2}\cdots\frac{d^2\mathbf{z}_{2k}}{r_c'^2}\exp(-\widetilde{S}')\\&\times\exp\Bigg\{-\frac{1}{2\pi}a_0'\ln(1+dl)\frac{1}{2}\sum_{\alpha\neq\beta}N_\alpha N_\beta\\&-\frac{1}{2\pi}\alpha_0'\ln(1+dl)\frac{1}{2}\sum_{\alpha\neq\beta}M_\alpha M_\beta\\&-\frac{i}{\pi}c_0'\ln(1+dl)\frac{1}{2}\sum_{\alpha\neq\beta}N_\alpha M_\beta\Bigg\},\end{aligned} \tag{A12}$$

where $\widetilde{S'}$ is the same as in Eq. (A3), but with r_c, G_N, G_M, and Γ replaced by r_c', G_N', G_M', and Γ', respectively. The second exponential is the correction from changing r_c to r_c' in the zero modes of the interactions.

Now, by charge neutrality $\Sigma N_\alpha=\Sigma M_\alpha=0$. Therefore, $\Sigma_{\alpha,\beta}M_\alpha N_\beta=0$. Also $\Sigma_{\alpha\neq\beta}N_\alpha N_\beta=(\Sigma N_\alpha)^2-\Sigma N_\alpha^2=-2j$ and similarly $\Sigma_{\alpha\neq\beta}M_\alpha M_\beta=-2k$, so that the last exponential in Eq. (A12) becomes $(1+dl)^{-2j(a_0'/4\pi)}(1+dl)^{-2k(\alpha_0'/4\pi)}$. With these contributions, the renormalization of y_N and y_M is

$$y_N'=y_N+y_N\left(2-\frac{a_0}{4\pi}\right)dl,$$

$$y_M'=y_M+y_M\left(2-\frac{\alpha_0}{4\pi}\right)dl \tag{A13}$$

to lowest order in y. At this point, the partition function is in the same form as in Eq. (A2), except that r_c has been replaced by r_c', and the Fourier coefficients and fugacities have changed according to Eqs. (A7) and (A13). This completes the renormalization-group program and gives us the the differential renormalization-group flow equations (17) stated in the text.

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