Teaching the Difference Between Stiffness and Damping

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A necessary step in the design of an effective control system is to understand its dynamics. It is not uncommon for a well-trained control engineer to use such an understanding to redesign the system to become easier to control, which requires an even deeper understanding of how the components of a system influence its overall dynamics. In this issue Henry C. Fu and Kam K. Leang discuss the difference between stiffness and damping when understanding the dynamics of mechanical systems.

T he simple spring, mass, and damper system is ubiquitous in dynamic systems and controls courses [1]. This column considers a concept students often have trouble with: the difference between a "soft" system, which has a small elastic restoring force, and a "damped" system, which dissipates energy quickly. In part, this confusion arises because softness and damping often come together in many everyday materials and systems. For example, probably the most obvious stiff (not soft) materials are metals, which are also very undamped, while highly damped plastics are much softer than metals.

This misconception is reinforced by the canonical example of a mass acted upon by a spring and a damper in parallel as illustrated in Figure 1(a). This classic parallel mass-spring-damper system is used to model the dynamics of piezoelectric actuators [2], [3], vehicle suspension systems [4], and gait-training equipment [5], for example. One way to measure the character of the response of such a system is to look at the Q factor of its resonance (see "The Q Factor"). For the parallel mass-spring-damper system, the Q factor at the resonant frequency is $Q \approx \sqrt{mk}/c$, where m is the mass, k is the spring constant, and c is the damping coefficient. Thus, increasing the spring constant k makes the behavior of the system more elastic and increases the Q factor, while decreasing the spring constant makes the system more damped and decreases the Q factor. However, stiffness and damping are not always correlated, and this column presents several examples to motivate a classroom discussion.

In contrast to the parallel spring-mass-damper, in the series mass-spring-damper system the stiffer the system, the more dissipative its behavior, and the softer the system, the more elastic its behavior. Thus teaching systems modeled by series mass-spring-damper systems allows students to appreciate the difference between stiffness and damping.

To get students to consider the difference between soft and damped, ask them to consider the following scenario: suppose a bullfrog is jumping to land on a very slippery floating lily pad as illustrated in Figure 2 and does not want to fall off. The frog intends to hit the flower (but cannot hold onto it) in the center to come to a stop. If the
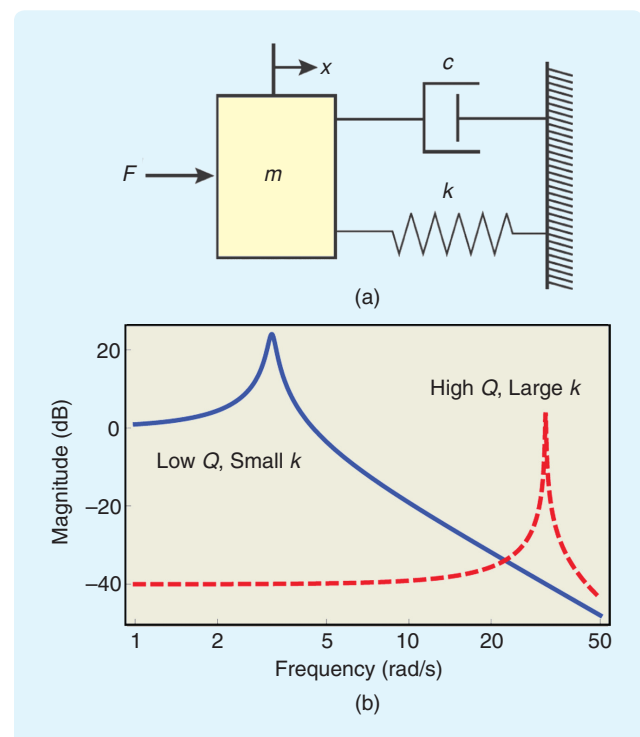

FIGURE 1 (a) The classic parallel mass-spring-damper system with mass m, damping constant c, and spring constant k [1]. (b) Response magnitude versus frequency for parallel spring-damper system with m = 0.1 kg and c = 0.02 kg/s. Solid line (blue): k = 1 kg/s^2; dashed line (red): k = 100 kg/s^2. For the parallel system, increasing the spring stiffness (k) increases the resonant frequency and leads to more elastic behavior, reflected in the larger Q factor.

flower (and stem) is modeled as a spring, does the frog want the spring to be stiff or soft? To guide the students' thinking, point out that the lily pad is free to move on the water surface but will feel a drag force, so it can be modeled as a damper (dashpot). Because the slippery lily pad does not exert lateral force on the frog, the entire system can be modeled as the mass-spring-damper in series, not parallel, as shown in Figure 3(a) with frog's mass m, spring constant k, and damping constant c. (This neglects the mass of the lily pad, which is justified as long as the frog is much heavier than the pad.)

What type of response does the frog want to see when it hits the flower at landing velocity? If the system has an elastic spring-like response, it will store the frog's energy and then return the energy, eventually pushing the frog backwards and off the slippery lily pad into the water. On the other hand, if the system has a damped response, it can dissipate the frog's energy so that the frog and the lily pad come to a stop together. Presented with this scenario, most students reply that to make the system damped the flower should be a soft spring, possibly due to their experience with materials and the parallel mass-spring-damper system in Figure 1(a).

However, the key to this example is that the series mass-spring-damper system offers more damping for a stiffer spring constant. To rationalize this, imagine that the spring is very soft—then upon impact, forces are low and the damper hardly moves while only
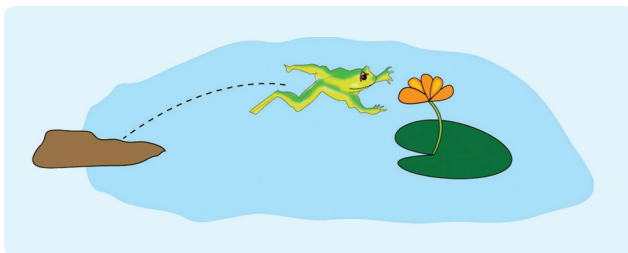
the spring deforms. Before the damper can dissipate energy, the spring rebounds and sends the frog back in the opposite direction. This example demonstrates that softness and damping need not go together; it is perfectly possible to have a stiff damped system as well as a soft undamped system.

Analyzing the Q factor of the resonant response of the series system further emphasizes that the system is less damped when soft. A glance at the frequency-dependent response for two different spring constants [Figure 3(b)] shows that increasing the spring constant k increases the resonance frequency with the familiar square root dependence on k but decreases the Q factor of the resonance. Thus, the stiffer the system, the more dissipative its behavior; while the softer the system, the more elastic its behavior. This behavior is completely opposite from the more familiar case of the parallel spring and damper system: for large Q factors, the series system has $Q \approx c/\sqrt{mk}$ while the parallel system has $Q \approx \sqrt{mk}/c$.

A student might ask: Is this model just a contrived example, or does it apply to any real systems? Viscoelastic fluids provide a real-life application. Viscoelastic behavior often results from polymeric solutions, in which the dissolved polymers behave like springs (Figure 4). Flows can stretch out the polymers, triggering an elastic response, but over a time constant determined by the viscous drag on the polymer and the elastic stiffness of the polymer molecules, the polymer relaxes back to it unstretched state and the
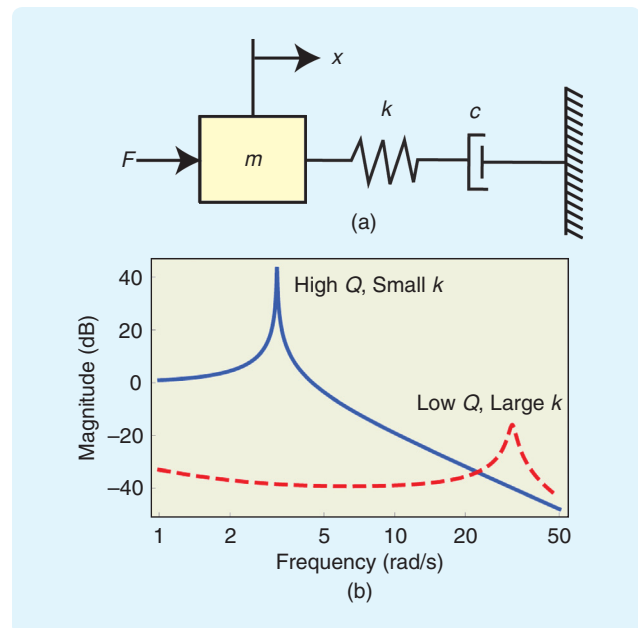
FIGURE 2 A bullfrog landing on a slippery lily pad will be stopped by the flower. Does the frog want the flower to be a soft or stiff spring?

FIGURE 3 (a) The bullfrog system can be modeled as a mass-spring-damper in series. (b) Response magnitude versus frequency for series spring-damper system with $m = 0.1$ kg and $c = 50$ kg/s. Solid line (blue): $k = 1$ kg/s^2; dashed line (red): $k = 100$ kg/s^2. Increasing the spring stiffness (k) leads to more damped behavior, reflected in the smaller Q factor.

elastic response fades. A common simple model of a viscoelastic fluid is the Maxwell constitutive equation, in which the (deviatoric) stress τ [6] obeys

$$\tau + \lambda \frac{\partial}{\partial t}\tau = -\eta\dot{\gamma}, \tag{1}$$

where λ is the time constant, η is the viscosity of the fluid, and $\dot{\gamma}$ is the strain rate in the fluid. For flows with a typical frequency ω, the behavior of a viscoelastic fluid is characterized by its "Deborah number," $\mathrm{De} = \omega\lambda$. For a high Deborah number, the fluid has elastic response, while for a small Deborah number, the fluid has viscous response. In the limit $\lambda \to 0$, $\mathrm{De} \to 0$ and the constitutive law reduces to that of a usual Newtonian fluid, for which the stress is given by product of viscosity and strain rate.

What does this example have to do with our series spring-damper system? Since the spring-like polymers are suspended in the fluid and have viscous damping drag, the Maxwell fluid can be represented by a spring and damper in series. Indeed, the time constant λ can be written in terms of the viscosity η and modulus G of the fluid as $\lambda = \eta/G$. For the same reasons as in our series model, the stiffer the system (larger G) the more it behaves as a fluid (small De), and the softer the system (smaller G) the more it behaves as an elastic solid (large De).

More insight into the series spring-damper model can be gained by constructing an electrical analogy consisting of a resistor and capacitor in parallel driven by a current source [Figure 5(a)]. For a sinusoidal current source with amplitude I_0 and frequency ω, the average power dissipated is plotted in Figure 5(b) as a function of the resistor resistance R. For a purely resistive circuit, it is expected that the power should be $I_0^2 R/2$. However, this behavior is only obtained for small R; for large R, there is much less power dissipated, and the dissipation decreases as the resistance R increases!

One way to understand this result is to observe that as the resistance R increases, more and more of the current flows through the capacitor. For high resistance, the behavior is dominated by the capacitance, and only for small R does the resistor see most of the current. This suggests a way to understand the series spring and damper in Figure 2(a). In that case, the total displacement of the mass is the sum of the spring and damper displacements. Just as increasing the resistance shunted more current to the capacitor, increasing the stiffness of the spring shunts more of the displacement to the damper, leading to behavior dominated by the damping element. In the limit of infinite spring constant, the spring is simply a rigid element and all the displacement is in the damper, so the response is completely damped.

Introducing the frog model shows students that softness and damping are independent properties of a system. Then, by examining a spring and damper in series, the students see how to model real systems that have the counter-
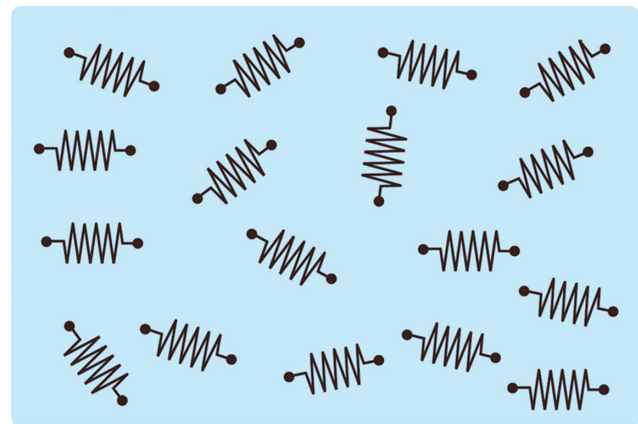

FIGURE 4 Elastic polymer molecules in solution lead to a fluid with viscoelastic response and can be modeled as a spring and damper in series.
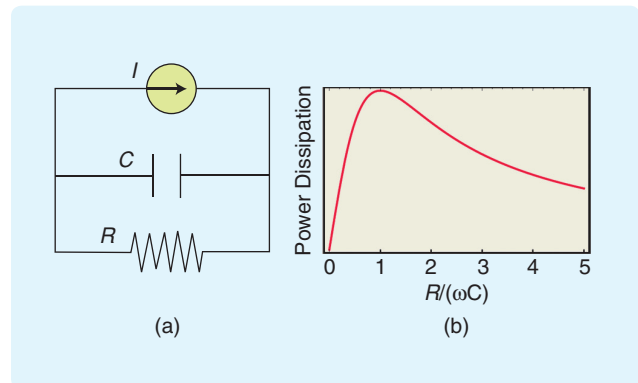

FIGURE 5 (a) Current driven resistor and capacitor in parallel. (b) Average power dissipation of the circuit is proportional to resistance only for small resistance R. Power dissipation decreases as resistance increases for large resistance.

intuitive combinations of being stiff and damped or soft and undamped. Including these examples in addition to the more familiar parallel spring and damper model prevents the accidental reinforcement of the misconceptions that stiffness and damping always go together, or are even the same thing.

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